# An Algorithmic Study of Relative Cardinalities for Interval-Valued Fuzzy Sets

Patryk Żywica*, Anna Stachowiak, Maciej Wygralak

*Department of Imprecise Information Processing Methods*
*Faculty of Mathematics and Computer Science, Adam Mickiewicz University*
*Umultowska 87, 61-614 Poznań, Poland*

**Abstract**

The main topic of this paper is the notion of relative cardinality for interval-valued fuzzy sets - its definition, properties and computation. First we define relative cardinality for interval-valued fuzzy sets following the concept of uncertainty modelling given by Mendel's Wavy-Slice Representation Theorem. We expand on previous approaches by considering relative cardinality based on different t-norms and scalar cardinalities and we initiate an investigation of its properties and possible applications. Drawing on the Nguyen–Kreinovich and Karnik–Mendel algorithms, we propose efficient algorithms to compute relative cardinality depending on a chosen t-norm. This seems to be the first such broad and consistent analysis to have been made of relative cardinality for interval-valued fuzzy sets. As a promising application we consider using interval-valued relative cardinality to construct the family of parameterised subsethood measures.

*Keywords:* relative scalar cardinality, interval-valued fuzzy sets, interval type-2 fuzzy sets, Karnik–Mendel algorithms, epistemic uncertainty, subsethood measure, inclusion index.
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## 1. Introduction

The need to model the imprecision and incompleteness of information has given rise to fuzzy set theory and its many extensions. Conventional fuzzy set theory is suitable for handling imprecise (gradual) statements, by allowing degrees of truth other than just true or false. However, it appears to be insufficient in the presence of incomplete (partial) information, when the exact degree

*Corresponding author
*Email addresses:* bikol@amu.edu.pl (Patryk Żywica), aniap@amu.edu.pl (Anna Stachowiak), wygralak@amu.edu.pl (Maciej Wygralak)

of truth cannot be specified. We recognise this kind of situation as involving uncertainty, the "real" truth value being concealed.

Undoubtedly, uncertainty is widespread in real life and practical applications, and cannot be ignored. The phenomenon has been studied for many years, and there are two main approaches to understanding the uncertainty of information [1] – epistemic and ontic. In both of them an uncertain concept is described by a set of its possible representations. In the epistemic approach, without additional knowledge it is impossible to pick the right one among them, although the right one exists. In the ontic one, all representations are equally acceptable and there is therefore no need to distinguish between them. For example let us consider a concept of a monthly salary for assistant professor, described by the interval [$4000, $6000]. Such an interval would have an ontic interpretation if it described the minimum and maximum threshold of the salary for the position of assistant professor in some institution. On the other hand, if this interval was supposed to describe an actual salary of some particular assistant professor, it should be interpreted in an epistemic way. What is important, both interpretations require different processing methods in order to capture the nature of uncertainty involved.

The fuzzy set community, motivated by the importance of uncertainty modelling and processing, has considered some generalisations of classical fuzzy set theory in order to capture the uncertainty factor. Of special interest among these are interval-valued fuzzy sets (IVFS, [2]), Atanassov's intuitionistic fuzzy sets (AIFS [3, 4]), interval type-2 fuzzy sets (IT2FS, [5, 6]) and general type-2 fuzzy sets (T2FS, [2, 7, 8]). All of these models have been widely applied in a variety of fields, including medical diagnosis [9, 10, 11], approximate reasoning [12, 13], classification [14], fuzzy control [15, 16], and decision making [17]. Since these approaches provide a more adequate representation of expert knowledge, they frequently outperform classical and type–1 fuzzy approaches. Characterisation of uncertainty associated with these concepts is a separate research problem [18]. In the following we adopt the IVFS approach; however, it should be noted that IVFS and AIFS are mathematically equivalent notions [19]. IVFS theory is able to represent both of the above-mentioned approaches to understanding uncertainty. The ontic one is more common in literature and, moreover, it is often adopted implicitly by the authors. The epistemic approach is still less explored but seems to better reflect many real-life problems and thus there is a need for research in this area. The results presented in this paper, although formally valid for both representations, have been developed especially for epistemic uncertainty.

Adding uncertainty to the field of consideration poses new challenges as regards to how to compare and operate on uncertain objects properly, effectively and without losing information about the amount of uncertainty. Much research has been done in this area, proving that these operations and relations are not just straightforward extensions of their crisp or fuzzy counterparts. Although for practical reasons some measures for uncertain objects are single values (see e.g. some similarity measures [20, 21, 22]), it seems to be more adequate to express those measures in an uncertain manner. Such an approach is employed, among

others, by Mendel's group, who use intervals to capture the uncertainty of an IT2FS. A deep study of uncertainty measures for IT2FS has been conducted, and centroid, cardinality, fuzziness, variance and skewness were all considered [23, 24]. It must be emphasised that, because of the more complex structure of an uncertain objects, the cost of computing uncertain measures is also higher, and thus the construction of such measures may be an algorithmic challenge. The issue of the effectiveness of computing such measures has been addressed by, among others, the Karnik–Mendel algorithms [25, 26]. However, there are still many important measures that require further research. The need for a general form of subsethood measure for IVFSs was the direct motivation of our study. Such a problem was also recognised by other researchers, see e.g. recent paper by Takáč [27], who constructed subsethood measures for interval-valued fuzzy sets based on the aggregation of interval fuzzy implications. Our paper is devoted to the approach proposed by Kosko [28], in which subsethood is defined in terms of the relative cardinality of two fuzzy sets. This idea is discussed in Section 3.

The primary objective of this paper is to extend relative cardinality of fuzzy sets to the IVFS case, and to construct effective algorithms for its computation. The notion of relative cardinality itself undoubtedly deserves much attention, as it provides a basis for many important concepts, not only the subsethood measure, but also similarity and entropy measures [29], implication operators [30], quantified sentences [31] and others. As a tool, it is widely applied e.g. in approximate reasoning [12], association rules quality assessment [32], rule-based systems [33], fuzzy control [34, 35], group decision making [36, 37], etc .

As a separate research problem, relative cardinality has never been given sufficient attention in the context of IVFSs. The need to fill this gap led us to construct t-norm-dependent interval-valued extensions of relative cardinality of fuzzy sets. As a tool for the extension we used the Wavy-Slice Representation Theorem [6], which has the desired ability to preserve the amount of uncertainty of IVFSs. To deal with the high complexity of the processed objects, we extended ideas from the Nguyen–Kreinovich [38] and Karnik–Mendel [25, 26] algorithms. We showed that the problem can be solved efficiently in the general case (for any t-norm). Never before has such a comprehensive analysis been made of relative cardinality for interval-valued fuzzy sets.

The rest of the paper is organised as follows. Section 2 gives some background information about fuzzy sets and IVFS. The third section covers relative cardinality, both in the fuzzy case as well as in the proposed extension to IVFS. In Section 4 we will address the problem of computing interval-valued relative cardinality. Section 5 contains an extensive evaluation of the proposed algorithms. Finally, in Section 6 we state some conclusions and offer areas for further research.

## 2. Definitions

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a crisp universal set. A mapping $A : X \to [0, 1]$ is called a fuzzy set in $X$. For each $1 \leq i \leq n$, the value $A(x_i)$ ($a_i$ for short)

represents the membership grade of $x_i$ in $A$. Let $\mathcal{F}$ be the family of all fuzzy sets in $X$.

A binary operation $t : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (t-norm, for short) if it is commutative, associative, non-decreasing in each argument, and has 1 as neutral element. The most important t-norms are minimum $t_{\min}(x,y) = \min(x,y)$, product $t_{\text{prod}}(x,y) = xy$, and Łukasiewicz $t_{\text{Łuk}}(x,y) = \max(0, x+y-1)$. A thorough investigation on t-norms is done in the classical monograph of Klement et.al. [39].

We consider fuzzy sets with basic set operations induced by a t-norm $t$. The intersection of two fuzzy sets $A, B \in \mathcal{F}$ is then defined as

$$\forall_{1 \le i \le n} (A \cap_t B)(x_i) = t(A(x_i), B(x_i)) \,. \tag{1}$$

Interval-valued fuzzy set theory, which is a special case of type-2 fuzzy set theory, was introduced by Zadeh [2]. Let $\mathcal{I}$ be the set of all closed subintervals of $[0,1]$. A mapping $\hat{A} : X \to \mathcal{I}$ is called an interval-valued fuzzy set. For each $1 \le i \le n$, the value $\hat{A}(x_i) = [\underline{A}(x_i), \overline{A}(x_i)] \in \mathcal{I}$ represents the membership of an element $x_i$ in $\hat{A}$. Usually $\underline{A}$ and $\overline{A}$ are called the lower and upper membership functions of $\hat{A}$ respectively. In epistemic approach, interval $\hat{A}(x_i)$ is understood to contain the true membership degree of $x_i$ in some incompletely known fuzzy set $A$ represented by $\hat{A}$. Its length reflects the amount of uncertainty about the membership degree of $i$-th element, taking values from 0 (when $\underline{A}(x_i) = \overline{A}(x_i)$) to 1 (when $[\underline{A}(x_i), \overline{A}(x_i)] = [0,1]$). We denote the set of all interval-valued fuzzy sets by $\mathcal{IVFS}$.

The cardinality of fuzzy sets has been extensively discussed in the literature (see [40]). In this paper we will focus on scalar cardinalities of fuzzy sets which can be characterised by the formula

$$\sigma_f(A) = \sum_{1 \le i \le n} f(A(x_i)) \,, \tag{2}$$

where $f : [0,1] \to [0,1]$ is a weighting function such that $f(0) = 0$, $f(1) = 1$ and $f(a) \le f(b)$ whenever $a \le b$. This approach formalises and reflects real human counting process under information imprecision [41]. The most common weighting function is the identity function $f_{\text{id}}(x) = x$. Consequently, cardinality of IVFS $\hat{A}$ is calculated as [42]

$$\hat{\sigma}_{f_1,f_2}(\hat{A}) = \left[\sigma_{f_1}(\underline{A}), \sigma_{f_2}(\overline{A})\right] \,, \tag{3}$$

where weighting functions $f_1$ and $f_2$ are such that $\forall_{1 \le i \le n}\ f_1(x_i) \le f_2(x_i)$. Most often it is assumed that $f_1 = f_2 = f_{\text{id}}$.

## 3. A concept of interval–valued relative cardinality of IVFSs

### *3.1. Relative cardinality of fuzzy sets*

Relative cardinality (or relative count) of fuzzy sets $A$ and $B$, introduced by Zadeh [43], represents the proportion of elements of $A$ which are in $B$. Using

the notation from the previous section, it can be written as

$$\sigma(A|B) = \frac{\sigma_{f_{\mathrm{id}}}(A \cap_{t_{\min}} B)}{\sigma_{f_{\mathrm{id}}}(B)} \,. \tag{4}$$

Relative cardinality was used by Zadeh as a simple and direct method for determining the truth degree of quantified sentences in natural language processing. Kosko [28] considered a subsethood measure derived from geometric interpretation which resulted in an idea identical to Zadeh's relative cardinality, although he does not use this term. Delgado et al. [44] carried out extensive studies on the evaluation of quantified sentences. A review of existing methods was conducted, and new ones based on relative cardinality were developed. All of these works considered relative cardinality in the sense of (4). The first who noticed the possibility of generalisation to other t-norms and scalar cardinalities was Wygralak [45]. A triangular-norm-based relative scalar cardinality (further referred as t-norm based relative cardinality), denoted by $\sigma_{f,t}(A|B)$ was defined as

$$\sigma_{f,t}(A|B) = \frac{\sigma_f(A \cap_t B)}{\sigma_f(B)} \text{ with } \sigma_f(B) \neq 0 \,, \tag{5}$$

where $t$ is an arbitrary t-norm and $f$ is a weighting function. Many important properties of t-norm based relative cardinality have been demonstrated [46]. The concept has been used as a tool in the Quantirius system for linguistic database summarisation [47].

The use of different t-norms in the relative cardinality formula leads to the construction of entirely new and diverse forms. Some of their properties and practical uses are considered in this section. Figures 1 and 2 present contour plots of the t-norm based relative cardinality $\sigma_{f,t}(A|B)$ of the triangular fuzzy set $A$ and fuzzy set $B$. A fuzzy set $A$ is indicated on the plot by a solid line. Membership values of elements of the fuzzy set $B$ range from 0 to 1, and for each case a value of $\sigma_{f,t}$ is calculated and represented in greyscale on the plot. The darker the colour, the higher the value of $\sigma_{f,t}(A|B)$ at that point.

Let us focus on Fig. 1. It can be seen that for standard relative cardinality with the minimum t-norm, all fuzzy sets $B$ with membership functions below $A$ obtain the maximum value of $\sigma_{f,t}(A|B)$ (the plot is totally black below $A$). This fully agrees with the intuitive interpretation of relative cardinality as a fuzzy subsethood measure. Different (and difficult to interpret) characteristics are obtained for the product t-norm, where the value of relative cardinality does not depend on $B$ in the whole domain. Finally, interesting results are obtained in case of the Łukasiewicz t-norm, where the relative cardinality increases with an increase in $B$'s membership value (quite opposite to the case of the minimum t-norm). This suggests that the subsethood interpretation is not intuitive for either the product or Łukasiewicz t-norms.

However, the most interesting result is that there are t-norms for which t-norm based relative cardinality has very similar characteristics to the minimum t-norm, but is more restrictive. Figure 2 shows contour plots for some Schweizer–Sklar t-norms with different values of the parameter $\lambda$. For $\lambda = -5$ the plot

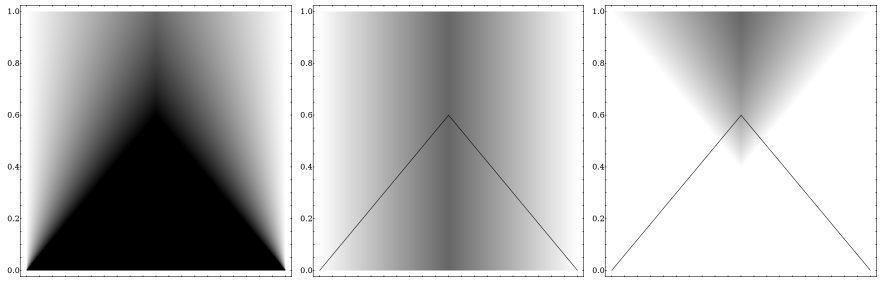

Figure 1: Contour plots of the relative cardinalities induced by the: minimum (left), product (middle) and Łukasiewicz (right) t-norms.

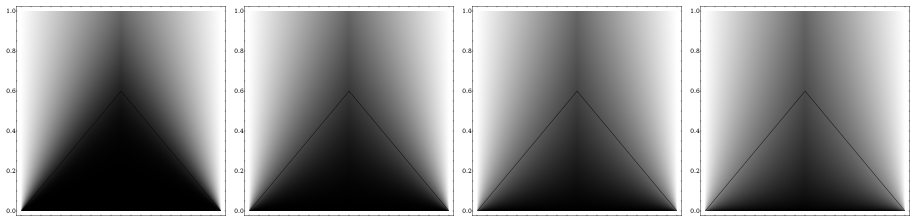

Figure 2: Contour plots of the relative cardinalities induced by Schweizer–Sklar t-norms with the $\lambda$ values: -5, -2, -1.25, -0.9 (from left to right).

is almost the same as for the minimum t-norm, but when $\lambda$ increases to 0 the characteristics of the relative cardinality gradually change, becoming more and more rigorous. This demonstrates the promising possibility of constructing parameterised subsethood measures with different properties. The applicability of similar parameterised conjunction operators was noticed e.g. in the context of fuzzy controllers in [48].

### *3.2. Interval–valued relative cardinality of IVFSs*

A number of studies aimed at extending the various measures, whether from the crisp to the fuzzy case, or from fuzzy to IVFS, have been conducted. Cornelis and Kerre showed that some measures extended to IVFS (they considered AIFS, to be exact) do not reduce to a mere double application of their fuzzy counterparts [49]. In the same work they recognised that resorting to interval calculus is not a viable option either.

The ability to model uncertainty in data is the key feature of IVFS. Thus the same feature should apply to the extended version of relative cardinality. A successful attempt to extend (5) to IVFS was made by Rickard et al. [50, 51] and Nguyen and Kreinovich [38]. They considered only the minimum t-norm and the identity weighting function. Their solution was based on Mendel's Wavy-Slice Representation Theorem for IT2FS [5, 6]. The present work adopts the same approach.

The main concept forming a basis for our extension formula is the Footprint

Of Uncertainty of an IVFS $\hat{A}$ (denoted by $FOU(\hat{A})$), defined as

$$\mathrm{FOU}(\hat{A}) = \{A \in \mathcal{F} \mid \forall_{1 \le i \le n}\ \underline{A}(x_i) \le A(x_i) \le \overline{A}(x_i)\}\,. \tag{6}$$

$FOU(\hat{A})$ is a set consisting of all fuzzy sets embedded in IVFS $\hat{A}$. Thus, it represents all values that can be hidden behind an uncertain $\hat{A}$. Whereas the classical IVFS representation draws attention only to the lower and upper bounds of a set, neglecting the variety of other fuzzy instantiations, $FOU(\hat{A})$ forces one to think of an IVFS $\hat{A}$ as an infinite set of fuzzy possibilities. A similar representation was also independently considered in the context of AIFSs by Stachowiak [52, 53]. Such an approach makes it possible to preserve the whole of the information about an incompletely known fuzzy set, and paves the way for a proper definition of operations on IVFSs, among others for a definition of relative cardinality given below.

**Definition 1.** The t-norm based interval–valued relative cardinality of two IVFSs $\hat{A}$ and $\hat{B}$, induced by a continuous t-norm $t$ and a continuous weighting function $f$, is defined as a function $\hat{\sigma}_{f,t} : \mathcal{IVFS} \times \mathcal{IVFS} \to \mathcal{I}$ given by

$$\hat{\sigma}_{f,t}(\hat{A}|\hat{B}) = \left\{\sigma_{f,t}(A|B) : A \in \mathrm{FOU}(\hat{A}), B \in \mathrm{FOU}(\hat{B})\right\} \tag{7}$$

$$= \left[\underline{\sigma}_{f,t}(\hat{A}|\hat{B}), \overline{\sigma}_{f,t}(\hat{A}|\hat{B})\right] \tag{8}$$

where

$$\underline{\sigma}_{f,t}(\hat{A}|\hat{B}) = \min_{\substack{A \in \mathrm{FOU}(\hat{A}) \\ B \in \mathrm{FOU}(\hat{B})}} \sigma_{f,t}(A|B)\,,$$

$$\overline{\sigma}_{f,t}(\hat{A}|\hat{B}) = \max_{\substack{A \in \mathrm{FOU}(\hat{A}) \\ B \in \mathrm{FOU}(\hat{B})}} \sigma_{f,t}(A|B)\,. \tag{9}$$

Note that, analogously as in (5), Definition 1 applies only when $\sigma_f(B) \neq 0$ for all $B \in \mathrm{FOU}(\hat{B})$. Hereafter we adopt this assumption. Moreover, the requirement for continuity of t-norm and a weighting function is necessary to ensure that the set defined by (7) is an interval.

Figure 3 shows an example IVFS $\hat{A}$ along with its FOU. Although embedded fuzzy sets can be of any shape, for simplicity we present two triangular ones, namely $A_1$ and $A_2$. We can easily calculate relative cardinalities:

$$\sigma_{f_{\mathrm{id}},t_{\min}}(A_1|A_2) = 1 \quad \text{and} \quad \sigma_{f_{\mathrm{id}},t_{\min}}(A_2|A_1) = 0.811\,. \tag{10}$$

However, computing the value of $\hat{\sigma}_{f_{\mathrm{id}},t_{\min}}(\hat{A}|\hat{A})$ requires much more effort. It is not sufficient to consider a single pair of embedded sets, but all possible pairs of such sets, regardless of their shape have to be taken into account. With use

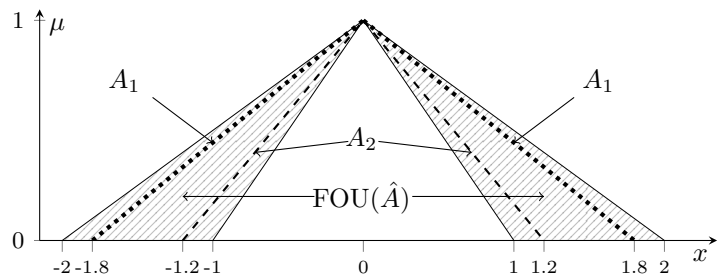


Figure 3: Visualisation of IVFS $\hat{A}$. Footprint Of Uncertainty (FOU) is indicated by shaded region. Dotted and dashed lines represent embedded sets $A_1$ and $A_2$, respectively.

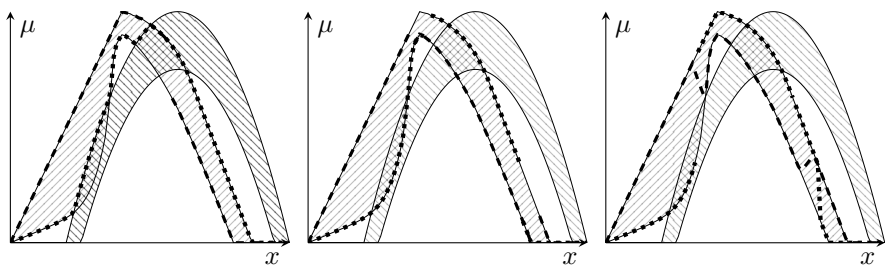


Figure 4: Visualisation of embedded fuzzy sets that reach the lower (dashed line) and upper (dotted line) bounds of the t-norm based interval-valued relative cardinalities induced by the minimum (left), product (middle) and Łukasiewicz (right) t-norms. Shaded regions represent Footprint Of Uncertainty (FOU) of input IVFSs.

of numerical optimisation we can calculate that

$$\hat{\sigma}_{f_{\mathrm{id}},t_{\min}}(\hat{A}|\hat{A}) = [0.614, 1] \ . \tag{11}$$

In general, interval $\hat{\sigma}_{f_{\mathrm{id}},t_{\min}}(\hat{A}|\hat{B})$ consists of all possible values obtained for any pair of embedded fuzzy sets of $\hat{A}$ and $\hat{B}$. The basic problem with determining the interval–valued relative cardinality defined in this way is finding the right $A \in \mathrm{FOU}(\hat{A})$ and $B \in \mathrm{FOU}(\hat{B})$ that minimise and maximise $\sigma_{f,t}(A|B)$. In general, these $A$ and $B$ are not simply equal to $\underline{A}, \overline{A}$ or $\underline{B}, \overline{B}$. Some possible shapes of $A$ and $B$, for $\hat{\sigma}_{f,t}$ with minimum, product and Łukasiewicz t-norms, are depicted in Fig. 4. Notice that $A$ and $B$ are not uniquely determined. For example, such a situation may occur for t-norms which attain a constant value over a range of arguments (such as nilpotent t-norms).

The next section presents the effective method of computing interval-valued relative cardinality parameterised with t-norms. The need and importance of considering different t-norms may be justified analogously as it was done in Section 3.1 since the Wavy-Slice Representation Theorem guarantees that the properties mentioned for fuzzy sets find their counterparts in the case of IVFSs.

## 4. Computing interval–valued relative cardinalities of IVFSs

The main problem with the interval-valued relative cardinality of IVFSs defined in Definition 1 is its computational cost. Formula (9) is computationally inefficient because $FOU(\hat{A})$ and $FOU(\hat{B})$ are infinite sets. In the general case, computing $\hat{\sigma}_{f,t}(\hat{A}|\hat{B})$ is equivalent to performing interval computations on non-linear functions, which is an NP-hard problem [54].

There are several papers which address this problem only in the cases of particular t-norm and the identity weighting function. The exhaustive algorithm presented by Rickard et al. for the minimum t-norm is not efficient [50]. Nguyen and Kreinovich [38] showed that computation of $\hat{\sigma}_{f,t}(\hat{A}|\hat{B})$ can be effective in the case of $f_{\mathrm{id}}$ and $t_{\min}$. They proposed algorithms for computing a lower bound of $\hat{\sigma}_{f_{\mathrm{id}},t_{\min}}(\hat{A}|\hat{B})$ in $O(n \cdot \log n)$ and an upper bound in $O(n)$ operations. Later, Wu and Mendel proposed another algorithm for computing lower bound with complexity $O(n^{1+\alpha})$, where $\alpha$ is a very small positive number [55]. An efficient solution for the case of the product t-norm has also been developed [56]. In this case, computing both the lower and upper bound of $\hat{\sigma}_{f_{\mathrm{id}},t_{\mathrm{prod}}}(\hat{A}|\hat{B})$ requires $O(n \cdot \log n)$ operations. We observed that the optimisation problem that must be solved to calculate the interval–valued relative cardinality for product t-norm is the same as the problem solved by Karnik–Mendel Algorithms for computing IT2FS centroid [25].

### *4.1. Problem reformulation*

Formula (9) can be rewritten in the following way:

$$\underline{\sigma}_{f,t}(\hat{A}|\hat{B}) = \min_{\substack{\underline{a}_i \le a_i \le \overline{a}_i \\ \underline{b}_i \le b_i \le \overline{b}_i}} \frac{\sum_{i=1}^{n} f(t(a_i, b_i))}{\sum_{i=1}^{n} f(b_i)} \tag{12}$$

and

$$\overline{\sigma}_{f,t}(\hat{A}|\hat{B}) = \max_{\substack{\underline{a}_i \le a_i \le \overline{a}_i \\ \underline{b}_i \le b_i \le \overline{b}_i}} \frac{\sum_{i=1}^{n} f(t(a_i, b_i))}{\sum_{i=1}^{n} f(b_i)} \,. \tag{13}$$

The arguments of min and max operators can be treated as a function of $n$ variables $b_1, \cdots, b_n$ which is non-decreasing with respect to $a_i$. For this reason (see also [38, 57]) the above formulas can be simplified to

$$\underline{\sigma}_{f,t}(\hat{A}|\hat{B}) = \min_{\underline{b}_i \le b_i \le \overline{b}_i} \frac{\sum_{i=1}^{n} f(t(\underline{a}_i, b_i))}{\sum_{i=1}^{n} f(b_i)} \tag{14}$$

and

$$\overline{\sigma}_{f,t}(\hat{A}|\hat{B}) = \max_{\underline{b}_i \le b_i \le \overline{b}_i} \frac{\sum_{i=1}^{n} f(t(\overline{a}_i, b_i))}{\sum_{i=1}^{n} f(b_i)} \,, \tag{15}$$

where $\forall_{1 \le i \le n}\ 0 \le \underline{b}_i \le \overline{b}_i \le 1$, and the t-norm $t$ as well as the weighting function $f$ are continuous. For simplicity, where it does not lead to ambiguity,

we denote $\underline{\sigma}_{f,t}(\hat{A}|\hat{B})$ and $\overline{\sigma}_{f,t}(\hat{A}|\hat{B})$ by $\underline{y}$ and $\overline{y}$. Such a problem definition will be used in the rest of the paper where the general question of calculation of the relative cardinality of IVFSs using other t-norms and weighting functions will be considered.

### *4.2. The Nguyen–Kreinovich and Karnik–Mendel algorithms*

The Nguyen–Kreinovich (NK, [38]) algorithms for $t_{\min}$ and $f_{\mathrm{id}}$ and Karnik–Mendel (KM, [25]) algorithms for $t_{\mathrm{prod}}$ and $f_{\mathrm{id}}$ will now be discussed in more detail. To shorten the discussion, only the algorithms for computing the lower bounds $\underline{\sigma}_{f,t}(\hat{A}|\hat{B})$ will be presented. The algorithms for upper bounds are similar and are based on the same approach. More details can be found in the source papers.

We want to show that these two algorithms, although meant for solving two different problems, and despite the significant differences in the original notation, generally share the same basis. This is the starting point for the construction of our algorithm, which is the main result presented in this paper. For both algorithms, we present pseudo-code taken from later works [26, 55]. This is because the algorithms were originally given only in a descriptive manner. For clarity, in both cases the pseudo-code omits most of the optimisation tricks proposed in both the original and later works. Moreover, the structure of the pseudo-code has been modified for ease of comparison. Both the algorithms are listed in Algorithm 1.

Although the NK and KM algorithms are very similar, there are some differences. We will discuss the most important ones. First, those two algorithms differ in iteration structure. Both of them execute the loop body until the current $\underline{y}$ value decreases. The NK algorithm performs a linear search for the point where a minimum is attained, and partial results are calculated for consecutive values of $k$. The KM algorithm is more similar to a binary search. It jumps between different values of $k$ in the search for the optimum. This difference causes a change in the average number of iterations. However, both algorithms require $O(n)$ iterations in the worst case. Each of these approaches can be converted to the other with little effort.

The next difference is the expression according to which the values $\underline{a}_i$, $\underline{b}_i$ and $\overline{b}_i$ are ordered. The NK algorithm uses

$$p_i^{\mathrm{NK}} = \frac{\min(\underline{a}_i, \overline{b}_i) - \min(\underline{a}_i, \underline{b}_i)}{\overline{b}_i - \underline{b}_i}\,, \tag{16}$$

while the KM algorithm uses $p_i^{\mathrm{KM}} = \underline{a}_i$, which is much simpler. These two expressions have a common source. Consider the following expression:

$$p_i^{*} = \frac{t(\underline{a}_i, \overline{b}_i) - t(\underline{a}_i, \underline{b}_i)}{\overline{b}_i - \underline{b}_i}\,, \tag{17}$$

which, in the case where $t$ is the product t-norm, reduces to $p_i^{\mathrm{KM}} = \underline{a}_i$.

**Algorithm 1** A pseudo-code of Nguyen and Kreinovich (left) and Karnik–Mendel (right) algorithms for computing $\underline{\sigma}_{f_{\mathrm{id}},t_{\mathrm{min}}}$ and $\underline{\sigma}_{f_{\mathrm{id}},t_{\mathrm{prod}}}$.

| | |
|---|---|
| Renumber $\underline{a}_i$, $\underline{b}_i$ and $\overline{b}_i$ so that $(\min(\underline{a}_i, \overline{b}_i) - \min(\underline{a}_i, \underline{b}_i))/(\overline{b}_i - \underline{b}_i)$ are sorted in ascending order | Renumber $\underline{a}_i$, $\underline{b}_i$ and $\overline{b}_i$ so that $\underline{a}_i$ are sorted in ascending order |
| **for** $1 \leq i \leq n$ **do** | **for** $1 \leq i \leq n$ **do** |
| $b_i \leftarrow \underline{b}_i$ | $b_i \leftarrow \frac{\underline{b}_i + \overline{b}_i}{2}$ |
| **end for** | **end for** |
| $\underline{y}' \leftarrow \frac{\sum_{i=1}^{n} \min(\underline{a}_i, b_i)}{\sum_{i=1}^{n} b_i}$ | $\underline{y}' \leftarrow \frac{\sum_{i=1}^{n} \underline{a}_i b_i}{\sum_{i=1}^{n} b_i}$ |
| $k \leftarrow 0$ | Find $k'$ such that $\underline{a}_{k'} < \underline{y} \leq \underline{a}_{k'+1}$ |
| **repeat** | **repeat** |
| $\underline{y} \leftarrow \underline{y}'$ | $\underline{y} \leftarrow \underline{y}'$ |
| $k \leftarrow k + 1$ | $k \leftarrow k'$ |
| | Find $k'$ such that $\underline{p}_{k'} < \underline{y} \leq \underline{p}_{k'+1}$ |
| $b_k \leftarrow \overline{b}_k$ | $b_i \leftarrow \begin{cases} \underline{b}_i & \text{if } i > k \\ \overline{b}_i & \text{if } i \leq k \end{cases}$ |
| $\underline{y}' \leftarrow \frac{\sum_{i=1}^{n} \min(\underline{a}_i, b_i)}{\sum_{i=1}^{n} b_i}$ | $\underline{y}' \leftarrow \frac{\sum_{i=1}^{n} \underline{a}_i b_i}{\sum_{i=1}^{n} b_i}$ |
| **until** $\underline{y}' < \underline{y}$ and $i \leq n$ | **until** $\underline{y}' < \underline{y}$ |
| **return** $\underline{y}$ | **return** $\underline{y}$ |

After renumbering, both the algorithms ensure that in each iteration

$$b_i = \begin{cases} \underline{b}_i & \text{if } i > k \\ \overline{b}_i & \text{if } i \leq k \end{cases} \tag{18}$$

for some $k$. This allows us to conclude that the optimal solution always has the following form

$$\underline{y} = \frac{\sum_{i=1}^{k} t(\underline{a}_i, \overline{b}_i) + \sum_{i=k+1}^{n} t(\underline{a}_i, \underline{b}_i)}{\sum_{i=1}^{k} \overline{b}_i + \sum_{i=k+1}^{n} \underline{b}_i}\,, \tag{19}$$

where $t$ is either the minimum or the product t-norm. From this equation, it can be seen that both algorithms are looking for the value of $k$ at which the result is optimal. Moreover, after minor optimisations, both algorithms have the same complexity $O(n \cdot \log n)$, where the most costly operation is the sorting in the first step.

### *4.3. Algorithms for t-norms with u-property*

The observation that similar algorithms solve the problem of t-norm based relative cardinality for two different t-norms was the main motivation to investigate the possibility of applying the same reasoning to a broader family of

t-norms. The aim was to solve the problem defined in (14) and (15), which is the most general version of t-norm based relative cardinality for IVFSs.

The first observation was the need to generalise the value of $p_i^*$. This plays a key role in the operation of both algorithms, indicating the order in which the elements of the domain are considered. It determines the impact of a particular item on the final result. Low values lead to a reduction in the final result, while high values cause it to increase. Therefore, in the case of minimisation, the smallest possible value of $p_i$ should be found. Similarly, in the case of maximisation, the largest possible value of $p_i$ should be found. This leads to the definition of the following two values:

$$\underline{p}_i = \inf_{\underline{b}_i < b \leq \overline{b}_i} \frac{f(t(\underline{a}_i, b)) - f(t(\underline{a}_i, \underline{b}_i))}{f(b) - f(\underline{b}_i)}, \tag{20}$$

and

$$\overline{p}_i = \sup_{\underline{b}_i < b \leq \overline{b}_i} \frac{f(t(\overline{a}_i, b)) - f(t(\overline{a}_i, \underline{b}_i))}{f(b) - f(\underline{b}_i)}. \tag{21}$$

Since the problems of minimising and maximising can be considered separately, without loss of generality it is assumed further that the elements $\underline{a}_i$, $\overline{a}_i$, $\underline{b}_i$ and $\overline{b}_i$ are ordered so that $\underline{p}_i \leq \underline{p}_{i+1}$ in the case of minimisation and $\overline{p}_i \geq \overline{p}_{i+1}$ in the case of maximisation. In this way, the domain elements that have the greatest impact on the final result are the first ones, regardless of the problem under consideration.

The next step was to generalise the property (19). The results are presented in the following theorems.

**Theorem 1.** *If a weighting function $f$ and t-norm $t$ are continuous, and $\underline{a}_i$, $\overline{a}_i$, $\underline{b}_i$ and $\overline{b}_i$ are ordered so that $\underline{p}_i \leq \underline{p}_{i+1}$, then there exists $\underline{k}$ and for each $i \leq \underline{k}$ there exists $\underline{u}_i$ such that $\underline{b}_i < \underline{u}_i \leq \overline{b}_i$ and*

$$\underline{y} = \frac{\sum_{i=1}^{\underline{k}} f(t(\underline{a}_i, \underline{u}_i)) + \sum_{i=\underline{k}+1}^{n} f(t(\underline{a}_i, \underline{b}_i))}{\sum_{i=1}^{\underline{k}} f(\underline{u}_i) + \sum_{i=\underline{k}+1}^{n} f(\underline{b}_i)}. \tag{22}$$

**Theorem 2.** *If the weighting function $f$ and t-norm $t$ are continuous, and $\underline{a}_i$, $\overline{a}_i$, $\underline{b}_i$ and $\overline{b}_i$ are ordered so that $\overline{p}_i \geq \overline{p}_{i+1}$, then there exists $\overline{k}$ and for each $i \leq \overline{k}$ there exist $\overline{u}_i$ such that $\underline{b}_i < \overline{u}_i \leq \overline{b}_i$ and*

$$\overline{y} = \frac{\sum_{i=1}^{\overline{k}} f(t(\overline{a}_i, \overline{u}_i)) + \sum_{i=\overline{k}+1}^{n} f(t(\overline{a}_i, \underline{b}_i))}{\sum_{i=1}^{\overline{k}} f(\overline{u}_i) + \sum_{i=\overline{k}+1}^{n} f(\underline{b}_i)}. \tag{23}$$

*Proofs are given in Appendix.*

These theorems reduce minimisation or maximisation to the problem of finding an optimal switch point $k$. This is a very similar result as in the case of the

NK and KM algorithms, but with one additional problem. These theorems do not provide any information on how to determine the values of $\underline{u}_i$ and $\overline{u}_i$. For $f_{\mathrm{id}}$ and $t_{\min}$ or $t_{\mathrm{prod}}$ we had $\underline{u}_i = \overline{b}_i$, as in the NK and KM algorithms discussed earlier. It should be noted that in general, finding these values exactly is nearly as difficult as solving the original problem. For this reason it is necessary to impose some restrictions on the t-norm and weighting function. Hence we introduce the concept of $u$-property, which significantly limits the variability of the values of $\underline{u}_i$ and $\overline{u}_i$. This, in turn, will help us to determine their values and allow them to be calculated.

**Definition 2.** A pair of functions $(t, f)$, where $t$ is a t–norm and $f$ is a scalar cardinality weighting function, has the $u$-property if for all $a, \underline{b}, \overline{b}$, $\underline{b} \leq \overline{b}$ there exist $\underline{u}$ and $\overline{u}$ such that

$$\forall_{0 \leq m \leq M} \ \arg\min_{\underline{b} < b \leq \overline{b}} \frac{m + f(t(a,b))}{M + f(b)} \in \{\underline{u}, \underline{b}\} \tag{24}$$

and

$$\forall_{0 \leq m \leq M} \ \arg\max_{\underline{b} < b \leq \overline{b}} \frac{m + f(t(a,b))}{M + f(b)} \in \{\overline{u}, \underline{b}\}\,. \tag{25}$$

Sometimes, for brevity, we will say that the t-norm $t$ has the $u$-property, by which we mean that the pair of functions $(t, f_{\mathrm{id}})$ has the $u$-property.

**Theorem 3.** *If the pair of functions $(t, f)$ has the $u$-property then*

$$\underline{u}_i = \arg\min_{\underline{b}_i < b \leq \overline{b}_i} \frac{1 + f(t(\underline{a}_i, b))}{1 + f(b)} \tag{26}$$

*and*

$$\overline{u}_i = \arg\max_{\underline{b}_i < b \leq \overline{b}_i} \frac{1 + f(t(\overline{a}_i, b))}{1 + f(b)}\,. \tag{27}$$

*Proof is given in Appendix.*

Theorem 3 allows us to make the first step towards calculation of the t-norm based interval–valued relative cardinality. Focusing on the class of pairs with the $u$-property allows a significant simplification while maintaining applicability to important practical problems. New algorithms will be presented for computing $\underline{y}$ and $\overline{y}$ in the case of t-norms and weighting functions with $u$-property. The algorithms use the values $\underline{p}_i$, $\overline{p}_i$, $\underline{u}_i$ and $\overline{u}_i$ defined earlier. These values can be computed numerically, or (for greater efficiency) precalculated for a specific t-norm and weighting function. In Table 1 we provide specific values of these parameters for popular t-norm families and weighting functions.

A procedure for computing $\underline{y}$ is given by Algorithm 2. An algorithm for computing $\overline{y}$ is very similar. There are only few differences. First of all, in the

**Algorithm 2** Algorithm for computing $\underline{y}$.

1: Renumber $\underline{a}_i$, $\underline{b}_i$ and $\overline{b}_i$ so that $\underline{p}_i$ are sorted in ascending order
2: $k' \leftarrow \lfloor n/2.4 \rfloor$
3: $m \leftarrow \sum_{i=1}^{k'} f(t(\underline{a}_i, \underline{u}_i)) + \sum_{i=k'+1}^{n} f(t(\underline{a}_i, \underline{b}_i))$
4: $M \leftarrow \sum_{i=1}^{k'} f(\underline{u}_i) + \sum_{i=k'+1}^{n} f(\underline{b}_i)$
5: $\underline{y} \leftarrow \frac{m}{M}$
6: **repeat**
7: $\quad k \leftarrow k'$
8: $\quad k' \leftarrow$ Find $k'$ such that $\underline{p}_{k'} < \underline{y} \leq \underline{p}_{k'+1}$
9: $\quad s \leftarrow \mathrm{sign}(k' - k)$
10: $\quad m \leftarrow m + s \sum_{i=\min(k,k')+1}^{\max(k,k')} (f(t(\underline{a}_i, \underline{u}_i)) - f(t(\underline{a}_i, \underline{b}_i)))$
11: $\quad M \leftarrow M + s \sum_{i=\min(k,k')+1}^{\max(k,k')} (f(\underline{u}_i) - f(\underline{b}_i))$
12: $\quad \underline{y} \leftarrow \frac{m}{M}$
13: **until** $k' \neq k$
14: **return** $\underline{y}$

case of $\overline{y}$, $\underline{p}_i$ and $\underline{a}_i$ are replaced by $\overline{p}_i$ and $\overline{a}_i$ respectively. Moreover, in the first step $\overline{p}_i$ are sorted in decreasing order and the direction of the inequalities in steps 1 and 8 is changed. Generally, both algorithms operate on the same principle as in the case of the KM algorithm (using the generalised values of $u_i$ and $p_i$). Neglecting the extra effort required for numerical optimisation (which should be avoided and replaced by precomputation where possible), both algorithms have the computational complexity $O(n \log n)$. The most expensive step is the first one, because it involves sorting. Afterwards the algorithm performs at most $n$ iterations, each with a fixed number of operations.

The following theorem provides much simpler conditions which ensure that a pair $(t, f)$ has the $u$-property.

**Theorem 4.** *A pair of functions $(t, f)$, where $t$ is a t–norm and $f$ is a weighting function, has the u-property if $f$ and $t$ are continuous, $f$ is strictly increasing, and for all $a$ and $\underline{b}$ there exist $q \geq \underline{b}$ and $\alpha > 0$ such that $t$ and $f$ are differentiable in $(q, 1]$ ($t$ in first variable, partial derivative is denoted by $t'_b$) and*

1.
$$\forall_{\underline{b} < b \leq q} \quad p_{a,\underline{b}}(b) = \frac{f(t(a,b)) - f(t(a,\underline{b}))}{f(b) - f(\underline{b})} = \alpha\,, \tag{28}$$

2.
$$\forall_{q \leq b \leq 1} \quad t'_b(a,b) = 0 \;\; or \;\; \frac{(f \circ t)'_b(a,b)}{f'(b)} \geq 1 \tag{29}$$

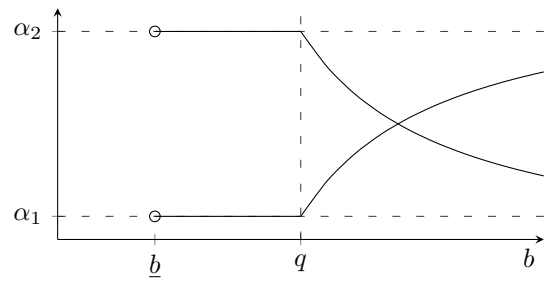


Figure 5: Two possible shapes of $p_{a,\underline{b}}$ for a pair $(t, f)$ satisfying Theorem 4.

*The proof is given in Appendix.*

The intuition behind this theorem in the case of the identity weighting function is that a t-norm for which $p_{a,\underline{b}}(b)$ is constant up to some point, and where beyond that point t-norm $t$ either increases rapidly enough or is constant, has the $u$-property. These two cases are presented in Fig. 5. Observe that

$$\underline{p}_i = \inf_{\underline{b}_i < b \leq \overline{b}_i} p_{\underline{a}_i, \underline{b}_i}(b)\,, \text{ and } \overline{p}_i = \sup_{\underline{b}_i < b \leq \overline{b}_i} p_{\overline{a}_i, \underline{b}_i}(b)\,. \tag{30}$$

Some $(t, f)$ pairs that fulfil the above conditions, and thus have $u$-property, are presented by the following examples.

**Example 1 (Minimum t-norm).** Let $f : [0, 1] \to [0, 1]$ be any continuous and increasing weighting function. First of all $p_{a,\underline{b}}$ needs to be computed; this is given by

$$p_{a,\underline{b}}(b) = \begin{cases} 1 & \text{if } \underline{b} < b \leq a \\ \frac{f(a)-f(\underline{b})}{f(b)-f(\underline{b})} & \text{if } \underline{b} < a < b \\ 0 & \text{if } a < \underline{b} < b \end{cases}. \tag{31}$$

To show the $u$-property we need to consider two possible cases. In the first one, if $a < \underline{b}$ then $p_{a,\underline{b}}(b)$ is constant and equals 0 ($\alpha = 0$ and $q = 1$). Note that there is no need to check the second condition, since there is no such $q < b \leq 1$. In the second case ($a \geq \underline{b}$), $\alpha = 1$ and $q = a$. A plot of $p_{a,\underline{b}}$ is given in Fig. 6. Moreover for $b > a$

$$[f(t_{\min}(a, b))]'_b = [f(t_{\min}(a, b)]'_b (t_{\min}(a, b))'_b = 0\,. \tag{32}$$

This reasoning leads to the conclusion that the pair $(t_{\min}, f)$ has the $u$-property for any continuous and increasing weighting function $f$.

Both algorithms require values of $\underline{p}_i$, $\overline{p}_i$, $\underline{u}_i$ and $\overline{u}_i$ which can be calculated directly (see Table 1). This result is more general than the original NK algorithm, because relative cardinality can be calculated for any continuous and strictly increasing weighing function.

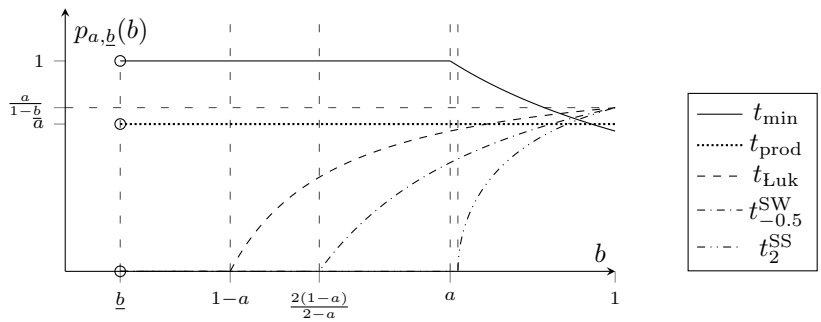


Figure 6: Shape of $p_{a,\underline{b}}$, depending on the t-norm used.

**Example 2 (Product t-norm).** The pair $(t_{\text{prod}}, f_{\text{id}})$ has $u$-property. In this case $p_{a,\underline{b}}(b) = a$, so it is constant, giving $\alpha = a$ and $q = 1$, which proves $u$-property. Moreover $\underline{p}_i = \underline{b}_i$, $\overline{p}_i = \overline{b}_i$, $\underline{u}_i = \overline{a}_i$ and $\overline{u}_i = \overline{a}_i$.

**Example 3 (Sugeno–Weber t-norms).** The Sugeno–Weber t-norm family is defined for each $\lambda > -1$ by the following formula

$$t^{SW}_{\lambda}(a,b) = \max\left(0, \frac{a+b-1+\lambda ab}{1+\lambda}\right). \tag{33}$$

To check whether a pair $(t^{SW}_{\lambda}, f_{\text{id}})$ has the $u$-property, we need to investigate the value of $p_{a,\underline{b}}(b)$ given by

$$p_{a,\underline{b}}(b) = \begin{cases} \frac{1+\lambda b}{1+\lambda} & \text{if } \frac{1-a}{1+\lambda a} < \underline{b} \\ 0 & \text{if } \underline{b} < b < \frac{1-a}{1+\lambda a} \\ 1 - \frac{(1-a)-\underline{b}(1+\lambda a)}{(1+\lambda)(b-\underline{b})} & \text{if } \underline{b} < \frac{1-a}{1+\lambda a} < b \end{cases}. \tag{34}$$

There are two possible cases. In the first one, when ${}^{(1-a)}/_{(1+\lambda a)} < \underline{b}$, the quotient is constant and equals ${}^{(1+\lambda a)}/_{(1+\lambda)}$ resulting in $\alpha = {}^{(1+\lambda a)}/_{(1+\lambda)}$ and $q = 1$. In the second case $({}^{(1-a)}/_{(1+\lambda a)} \geq \underline{b})$, $\alpha = 0$ and $q = {}^{(1-a)}/_{(1+\lambda a)}$. Moreover for $b > {}^{(1-a)}/_{(1+\lambda a)}$

$$t^{\text{SW}\prime}_{\lambda\ b}(a,b) = \left(\frac{a+b-1+\lambda ab}{1+\lambda}\right)'_b = \frac{1+\lambda a}{1+\lambda}. \tag{35}$$

Because ${}^{(1+\lambda a)}/_{(1+\lambda)} > 1$ only if $\lambda < 0$, this proves that $(t^{SW}_{\lambda}, id)$ has $u$-property when $-1 < \lambda \leq 0$. Observe that for $\lambda = 0$ $t^{SW}_0$ becomes a Łukasiewicz t-norm which for this reason also has the $u$-property.

Table 1 lists the parameters required by both algorithms in the case of the identity weighting function, for the most common t-norms. Figure 6 contains plots of $p_{a,\underline{b}}$ for those t-norms.

| t-norm | | $c_i^\lambda$ | $c_i^\lambda < \underline{a}_i$ | $\underline{a}_i \le c_i^\lambda < \overline{a}_i$ | $\overline{a}_i \le c_i^\lambda$ |
|---|---|---|---|---|---|
| Minimum | $\underline{p}_i$ | $\underline{b}_i$ | 0 | $\frac{\underline{b}_i - \underline{a}_i}{\overline{a}_i - \underline{a}_i}$ | 1 |
| | $\underline{u}_i$ | | $\overline{a}_i$ | $\overline{a}_i$ | $\overline{a}_i$ |
| $t_{\min}$ | $\overline{p}_i$ | $\overline{b}_i$ | 0 | 1 | 1 |
| | $\overline{u}_i$ | | $\underline{a}_i$ | $\overline{b}_i$ | $\overline{a}_i$ |
| Product | $\underline{p}_i$ | - | | $\underline{b}_i$ | |
| | $\underline{u}_i$ | | | $\overline{a}_i$ | |
| $t_{\text{prod}}$ | $\overline{p}_i$ | - | | $\overline{b}_i$ | |
| | $\overline{u}_i$ | | | $\overline{a}_i$ | |
| Łukasiewicz | $\underline{p}_i$ | $1 - \underline{b}_i$ | 1 | 0 | 0 |
| | $\underline{u}_i$ | | $\overline{a}_i$ | $1 - \underline{b}_i$ | $\overline{a}_i$ |
| $t_{\text{Łuk}}$ | $\overline{p}_i$ | $1 - \overline{b}_i$ | 1 | $\frac{\overline{a}_i + \overline{b}_i - 1}{\overline{a}_i - \underline{a}_i}$ | 0 |
| | $\overline{u}_i$ | | $\overline{a}_i$ | $\overline{a}_i$ | $\underline{a}_i$ |
| Sugeno–Weber | $\underline{p}_i$ | $\frac{1-\underline{b}_i}{1+\lambda \underline{b}_i}$ | $\frac{1+\lambda \underline{b}_i}{1+\lambda}$ | 0 | 0 |
| | $\underline{u}_i$ | | $\overline{a}_i$ | $\frac{1-\underline{b}_i}{1+\lambda \underline{b}_i}$ | $\overline{a}_i$ |
| $t_\lambda^{\text{SW}}$, $\lambda \in (-1, 0)$ | $\overline{p}_i$ | $\frac{1-\overline{b}_i}{1+\lambda \overline{b}_i}$ | $\frac{1+\lambda \overline{b}_i}{1+\lambda}$ | $\frac{t_\lambda^{\text{SW}}(\overline{a}_i, \overline{b}_i)}{\overline{a}_i - \underline{a}_i}$ | 0 |
| | $\overline{u}_i$ | | $\overline{a}_i$ | $\overline{a}_i$ | $\underline{a}_i$ |
| Schweizer–Sklar | $\underline{p}_i$ | $\sqrt[\lambda]{1 - \underline{b}_i^\lambda}$ | 1 | 0 | 0 |
| | $\underline{u}_i$ | | $\underline{a}_i$ | $\sqrt[\lambda]{1 - \underline{b}_i^\lambda}$ | $\overline{a}_i$ |
| $t_\lambda^{\text{SS}}$, $\lambda \ge 1$ | $\overline{p}_i$ | $\sqrt[\lambda]{1 - \overline{b}_i^\lambda}$ | 1 | $\frac{\sqrt[\lambda]{\overline{a}_i^\lambda + \overline{b}_i^\lambda - 1}}{\overline{a}_i - \underline{a}_i}$ | 0 |
| | $\overline{u}_i$ | | $\overline{a}_i$ | $\overline{a}_i$ | $\underline{a}_i$ |

Table 1: Parameters required by the algorithms for the most common t-norms.

It may be noted that in some cases $\underline{p}_i$ or $\overline{p}_i$ attains only two values: 0 or 1. For example, this occurs in the case of the minimum t-norm for the upper bound. Nguyen showed that in this situation it is possible to calculate the relative cardinality directly without sorting [38]. Similar reasoning can be applied to the t-norm based relative cardinality when $\underline{p}_i$ or $\overline{p}_i$ takes only the two extreme values.

### *4.4. General solution for t-norms without u-property*

Unfortunately, not all t-norms have the $u$-property, and so further investigation is required into the problem of calculating the t-norm based relative cardinality in this case. The $u$-property imposes a strict constraint on the values of $\underline{u}_i$ and $\overline{u}_i$ from Theorems 1 and 2, making it possible to treat individual values of $\underline{u}_i$ and $\overline{u}_i$ independently. If a t-norm does not have $u$-property, the

problem of finding $\underline{u}_i$ and $\overline{u}_i$, and thus the exact value of the relative cardinality, becomes much more harder.

Theorems 1 and 2 guarantee the existence of $\underline{u}_i$ and $\overline{u}_i$. Let us now examine their properties more closely. For each $i \leq k$, the $i$th element may be taken outside the sum in following way

$$\underline{y} = \frac{f(t(\underline{a}_i, \underline{u}_i)) + \sum_{\substack{j=1 \\ i \neq j}}^{\underline{k}} f(t(\underline{a}_j, \underline{u}_j)) + \sum_{j=\underline{k}+1}^{n} f(t(\underline{a}_j, \underline{b}_j))}{f(\underline{u}_i) + \sum_{\substack{j=1 \\ i \neq j}}^{\underline{k}} f(\underline{u}_j) + \sum_{j=\underline{k}+1}^{n} f(\underline{b}_j)} . \tag{36}$$

Because $\underline{y}$ minimises the whole expression, we can observe that

$$\underline{u}_i = \underset{\underline{b}_i < b \leq \overline{b}_i}{\arg\min} \frac{f(t(\underline{a}_i, b)) + \sum_{\substack{j=1 \\ i \neq j}}^{\underline{k}} f(t(\underline{a}_j, \underline{u}_j)) + \sum_{j=\underline{k}+1}^{n} f(t(\underline{a}_j, \underline{b}_j))}{f(b) + \sum_{\substack{j=1 \\ i \neq j}}^{\underline{k}} f(\underline{u}_j) + \sum_{j=\underline{k}+1}^{n} f(\underline{b}_j)} . \tag{37}$$

Thus, each $\underline{u}_i$ depends on all the other $\underline{u}_j$.

The proposed approach to this problem involves the construction of a recursive equation for the value $\underline{u}_i$ and an iterative algorithm that will approximate it. The equation is easily derived from (37)

$$\underline{u}_i^{(l+1)} = \underset{\underline{b}_i < b \leq \overline{b}_i}{\arg\min} \frac{f(t(\underline{a}_i, b)) + \sum_{\substack{j=1 \\ i \neq j}}^{\underline{k}} f(t(\underline{a}_j, \underline{u}_j^{(l)})) + \sum_{j=\underline{k}+1}^{n} f(t(\underline{a}_j, \underline{b}_j))}{f(b) + \sum_{\substack{j=1 \\ i \neq j}}^{\underline{k}} f(\underline{u}_j^{(l)}) + \sum_{j=\underline{k}+1}^{n} f(\underline{b}_j)} , \tag{38}$$

with initial values

$$\underline{u}_i^{(0)} = \begin{cases} \underline{b}_i & \text{if } i > k \\ \overline{b}_i & \text{if } i \leq k \end{cases} . \tag{39}$$

It should be noted that in this equation $\underline{u}_i^{(l+1)}$ does not depend directly on $\underline{u}_i^{(l)}$. However, there is an indirect dependence on $\underline{u}_i^{(l-1)}$ through other $\underline{u}_j^{(l)}$. The use of this recursive equation does not guarantee that the correct value of $\underline{u}_i$ will be obtained. However, as has been shown in an extensive evaluation, it converges very quickly to the optimal solution. Of course, the above reasoning can also be applied for $\overline{u}_i$, giving analogous results.

The main goal when designing the algorithm for computing t-norm based relative cardinality for t-norms without $u$-property was to integrate the recursive equation within the algorithms proposed in the previous section. The basic principle is to execute Algorithm 2 iteratively until the change in the resulting value is negligibly small (less than a given $\varepsilon$). The pseudo-code of the algorithm for computing lower bound is given in Algorithm 3. As in the previous section, algorithm for computing $\overline{y}$ is very similar.

**Algorithm 3** Approximate algorithm for computing $\underline{y}$.

Renumber $\underline{a}_i$, $\underline{b}_i$ and $\overline{b}_i$ so that $\underline{p}_i$ are sorted in ascending order
$k' \leftarrow \lfloor n/2.4 \rfloor$
$m \leftarrow \sum_{i=1}^{k'} f(t(\underline{a}_i, \underline{u}_i)) + \sum_{i=k'+1}^{n} f(t(\underline{a}_i, \underline{b}_i))$
$M \leftarrow \sum_{i=1}^{k'} f(\underline{u}_i) + \sum_{i=k'+1}^{n} f(\underline{b}_i)$
$y' \leftarrow \frac{m}{M}$
**repeat**
  $\underline{y} \leftarrow \frac{m}{M}$
  **repeat**
    $k \leftarrow k'$
    $k' \leftarrow$ Find $k'$ such that $\underline{p}_{k'} < \underline{y} \leq \underline{p}_{k'+1}$
    $s \leftarrow \mathrm{sign}(k' - k)$
    $m \leftarrow m + s \sum_{i=\min(k,k')+1}^{\max(k,k')} (f(t(\underline{a}_i, \underline{u}_i)) - f(t(\underline{a}_i, \underline{b}_i)))$
    $M \leftarrow M + s \sum_{i=\min(k,k')+1}^{\max(k,k')} (f(\underline{u}_i) - f(\underline{b}_i))$
    $\underline{y} \leftarrow \frac{m}{M}$
  **until** $k' \neq k$
  $e \leftarrow |y' - \underline{y}|$
  $y' \leftarrow \underline{y}$
**until** $e < \varepsilon$
**return** $\underline{y}$

## 5. Evaluation

The proposed approximate algorithms were subjected to extensive evaluation. Five test sets for different domain sizes ($n = 3, 5, 10, 25, 100$) were used. Each of them consisted of 480000 pairs of different interval-valued fuzzy sets. It was ensured that each test set contained IVFSs with different shapes of membership function (Trapezoid, Gaussian, Bell, Sigmoid, and generated randomly) in the same proportions. Moreover, different degrees of intersection of those IVFSs were chosen.

Both algorithms were implemented in Java. Due to the finite numerical accuracy, some programming-language-specific optimisations were applied. The values of $\underline{p}_i$, $\overline{p}_i$ and $\underline{u}_i$, $\overline{u}_i$ used in the algorithms were calculated numerically using the Brent method [58]. In addition, it was necessary to add protection against an infinite loop caused by inaccurate values of $\underline{p}_i$ or $\overline{p}_i$. IVFS pairs for which this problem occurred were excluded from the test set. The algorithm implementation as well as test script is publicly available at `http://min.wmi.amu.edu.pl/en`.

The test involved the Dombi, Frank, Hamacher, Schweizer–Sklar, Sugeno–

Weber and Yager t-norm families with different values of parameters. Three basic t-norms – minimum, product and Łukasiewicz – were also directly included in the test. All t-norm based relative cardinalities were computed using the identity weighting function and $\varepsilon = 10^{-6}$. To enable comparison of the results, all of the relative cardinalities (also for the t-norms with $u$-property, for which precalculation is possible) were computed using the same algorithm implementation based on numerical optimisation. No optimisation specific to any particular t-norm was used. For each t-norm, separately for the lower and upper bound, the average and maximum running time in milliseconds, the number of main loop iterations and the number of inner loop iterations were measured. Detailed results are presented in supplementary materials attached to this paper. Here we will only summarise them.

For all t-norms with the $u$-property, both algorithms perform at most two main loop iterations. This means that in this case the approximate algorithms reduce to those for t-norms with the $u$-property. This is a very important characteristic, which makes it possible to use the approximate algorithms in all cases, even when the $u$-property condition is met, without any substantial decrease in performance. This can be easily observed in the case of Schweizer–Sklar t-norms. The number of iterations necessary to compute the lower bound of the relative cardinality in this case varies significantly depending on whether $\lambda$ is smaller or larger than 1 (for $\lambda \geq 1$ this t-norm has the $u$-property).

Generally, an average number of iterations is very small for both algorithms. There is no t-norm which on average required more than 3 iterations of the outer loop and 5 of inner loop. It seems that the calculation of the upper bound is somewhat simpler. However, no reason was found for this. Average computing time differs depending on the t-norm (from 3.42ms for product t-norm and 4.49ms for t-norm minimum to nearly 100ms for some Yager t-norms), despite a similar number of iterations. This is due to the computational effort associated with Brent optimisation. Some t-norms are much simpler to optimise than others. The maximum number of iterations for the majority of t-norms also did not differ significantly from the average. An interesting observation is that for certain t-norms it is easier to determine the upper or lower bound. This is especially evident in the case of the maximum number of iterations for the Hamacher and Frank t-norms. For $\lambda < 1$ computing the upper bound is considerably easier than the lower, while for $\lambda > 1$ computing the lower bound requires fewer iterations than the upper.

In some cases the algorithm converges even in the first iteration, which may be confusing. This happens when the first iteration does not improve the solution, which is possible for some IVFSs and t-norms. There are two cases when this can happen. First, the initial value of $k$ may be the optimal one. In addition, for some t-norms (especially nilpotent ones) it is possible that no matter which point is chosen from the interval $[\underline{b}_i, \overline{b}_i]$, the resulting value of the relative cardinality does not change significantly. For example, the Łukasiewicz t-norm may be zero even if one of the arguments is greater than 0.5, which may lead to such behaviour.

As has been observed by Wu [26], proper setting of the initial value of $k$

can significantly affect the number of inner iterations needed to achieve the final result. Wu and Nie [59] in their paper provide better estimate for the initial value for the KM algorithm. They indicated $n/2.4$ as the optimum initial value in the case of the product t-norm. The average value of the final $k$ in our evaluation is 41.92 (for $n = 100$), which gives $n/2.39$ as the optimal initial $k$ for the product t-norm. This very similar value confirms the methodology of evaluation. However, it turns out, that in the case of a relative cardinality it is difficult to develop a initialisation strategy effective for all t-norms. Generally, the average final value of $k$ should be used to calculate the optimal initial value of $k$. We highly recommend to adjust $k'$ value according to used t-norm and its parameters either by computer simulation or by using values gathered in the supplement. Moreover it can be observed from the evaluation results that average final $k$ values are sometimes very different for the lower and upper bounds of relative cardinality. Thus it is reasonable to consider separate initial values for the two algorithms.

## 6. Conclusions and further research

The main contribution of the present work is the set of algorithms for efficient calculation of the t-norm based relative cardinalities of IVFSs, given in Section 4. We considered the most important classes of t-norms, concluding that the $u$-property determines which algorithm should be chosen to solve a given problem. Our approach fully coincides with earlier solutions given by the Nguyen–Kreinovich and Karnik–Mendel algorithms, which are special cases of our algorithms. We believe that our systematic and comprehensive algorithmic study has thrown new light on the still open problem of effectiveness of calculation on IVFSs. A still outstanding task is to make a thorough mathematical analysis of approximate algorithms. A very good starting point for such research is the recent paper by Han and Liu that shows the global convergence of the KM algorithm [60]. Moreover, some results of the evaluation of the algorithms suggest the possibility of a stronger version of the $u$-property.

Since the relative cardinality of IVFSs can be effectively obtained, the next step is to explore areas of its practical application and possible interpretations. Although the classical relative cardinality based on the minimum t-norm can serve as a subsethood measure, this appears not to be true for some other t-norms. For example, for the product and Łukasiewicz t-norms, we probably need to look for another interpretation of t-norm based relative cardinality, and this must be done for both fuzzy and interval-valued fuzzy sets. Since relative cardinality was originally based on conditional probability, the problem of t-norm selection may be approached similarly as in the recent work of Yager [61]. We have initiated research on this topic by noting that a t-norm based relative cardinality with Schweizer–Sklar t-norms has an interesting property of being more restrictive than that based on the minimum t-norm. This suggests that t-norm based relative cardinality can be used to obtain a family of parameterised subsethood measures with features that are desirable for some practical problems. The first premise in this direction is the use of proposed algorithms

in ovarian tumor diagnosis support system [9, 14]. The system consists of classification module with similarity measure based on IVFS relative cardinality. Further applications are the subject of our current research. It should also be noted that the results obtained for IVFSs may also be valid for general type-2 fuzzy sets, as has been shown in the case of the Karnik–Mendel algorithms [62].

## Appendix A.

Proof of Theorems 1 and 2. Proofs of Theorems 1 and 2 are analogous so, we will show only the first one. We will prove it in two steps. First we will prove the existence of some points $b_i^*$ at which

$$\underline{y} = \frac{\sum_{i=1}^n f(t(\underline{a}_i, b_i^*))}{\sum_{i=1}^n f(b_i^*)} \,. \tag{A.1}$$

Then we will show that under the assumptions of the theorem, the required conditions are met by those values.

As was stated in the main part of the paper, $\underline{y} = \frac{\sum_{i=1}^n f(t(a_i,b_i)}{\sum_{i=1}^n f(b_i)}$ can be treated as a function of $n$ variables $b_1, \cdots, b_n$ which is non-decreasing with respect to $a_i$. For this reason (see also [57]) the formula for $\underline{y}$ can be simplified:

$$\underline{y} = \min_{\forall b_i \in [\underline{b_i}, \overline{b_i}]} \frac{\sum_{i=1}^n f(t(\underline{a}_i, b_i))}{\sum_{i=1}^n f(b_i)} \,. \tag{A.2}$$

Moreover, this function is continuous except in the degenerated case when $f(\underline{b}_i) = 0$ for all $1 \leq i \leq n$. Thus, as for any continuous function on a compact set, the global minimum $\underline{y}$ is attained for some $b_1^*, \cdots, b_n^*$. Therefore the existence of these points has been proved.

We define the projection of $\underline{y}$ on the $i$th variable $r_i : [\underline{b}_i, 1] \to [0, 1]$

$$r_i(b_i) = \frac{\sum\limits_{j \neq i}^n f(t(\underline{a}_j, b_j^*)) + f(t(\underline{a}_i, b_i))}{\sum\limits_{j \neq i}^n f(b_j^*) + f(b_i)} = \frac{m_i + f(t(\underline{a}_i, b_i))}{M_i + f(b_i)} \tag{A.3}$$

Next we try to determine when $b_i^* > \underline{b}_i$. This happens whenever $r_i(b_i^*) < r_i(\underline{b}_i)$, which after transformation is equivalent to

$$\frac{f(t(\underline{a}_i, b_i^*)) - f(t(\underline{a}_i, \underline{b}_i)}{f(b_i^*) - f(\underline{b}_i)} < \frac{f(t(\underline{a}_i, \underline{b}_i)) + m_i}{f(\underline{b}_i) + M_i} \,. \tag{A.4}$$

Note that (A.4) is satisfied if and only if $\underline{p}_i < \underline{y}$. For this reason values of $\underline{p}_i$ greater than 1 can be truncated since $\underline{y}$ is always less than or equal to 1.

From these observations, we can conclude that $b_i^* > \underline{b}_i$ only if $\underline{p}_i < \underline{y}$, otherwise $b_i^* = \underline{b}_i$. Assuming that the $\underline{p}_i$ are arranged in ascending order, i.e.

$\underline{p}_1 \leq \underline{p}_2 \leq \cdots \leq \underline{p}_n$, there exists $k$ such that $\underline{y}$ can be expressed in the following way:

$$\underline{y} = \frac{\sum_{i=1}^{k} f(t(\underline{a}_i, b_i^*)) + \sum_{i=k+1}^{n} f(t(\underline{a}_i, \underline{b}_i))}{\sum_{i=1}^{k} f(b_i^*) + \sum_{i=k+1}^{n} f(\underline{b}_i)} \,. \tag{A.5}$$

Indeed, there are three possible cases and in each of them it is possible to determine the value of $k$:

- $k = 0$, whenever $\underline{y} < \underline{p}_1$,
- $k = j$, whenever $\underline{p}_j < \underline{y} \leq \underline{p}_{j+1}$ for some $j$,
- $k = n$, whenever $\underline{p}_n < \underline{y}$.

To complete the proof, we define $\underline{u}_i$ for $i \leq k$ as follows: $\underline{u}_i = b_i^*$.

Proof of Theorem 3. This theorem follows from a direct application of the projection from proof of Theorems 1 and 2 to the Definition 2. For simplicity only the first part will be shown. Let $k$ and $u_i$ be as in Theorem 1. Using the projection from (A.3), it is easy to see that for each $i \leq k$

$$\underline{y} = r_i(\underline{u}_i) = \frac{m_i + f(t(\underline{a}_i, \underline{u}_i))}{M_i + f(\underline{u}_i)} \,, \tag{A.6}$$

where $m_i = \sum_{i \neq j}^{k} f(t(\underline{a}_j, \underline{u}_j)) + \sum_{j=k+1}^{n} f(t(\underline{a}_j, \underline{b}_j)$ and $M_i = \sum_{i \neq j}^{k} f(\underline{u}_j) + \sum_{j=k+1}^{n} f(\underline{b}_j)$. Thus, $\underline{u}_i$ is the value that minimises this expression, so it is possible to write

$$\underline{y} = \min_{\underline{b}_i < b \leq \overline{b}_i} r_i(b) = \min_{\underline{b}_i < b \leq \overline{b}_i} \frac{m_i + f(t(\underline{a}_i, b))}{M_i + f(b)} \,. \tag{A.7}$$

This in turn gives that

$$\underline{u}_i = \arg\min_{\underline{b}_i < b \leq \overline{b}_i} \frac{m_i + f(t(\underline{a}_i, b))}{M_i + f(b)} \,. \tag{A.8}$$

Now it is possible to use the $u$-property, which states that

$$\arg\min_{\underline{b}_i < b \leq \overline{b}_i} \frac{m_i + f(t(\underline{a}_i, b))}{M_i + f(b)} = \arg\min_{\underline{b}_i < b \leq \overline{b}_i} \frac{1 + f(t(\underline{a}_i, b))}{1 + f(b)} \,, \tag{A.9}$$

and hence

$$\underline{u}_i = \arg\min_{\underline{b}_i < b \leq \overline{b}_i} \frac{1 + f(t(\underline{a}_i, b))}{1 + f(b)} \,. \tag{A.10}$$

Proof of Theorem 4. Let $a$, $\underline{b}$, $q$ and $\alpha$ be as in Theorem 4, and let $\overline{b}$ be any number such that $\underline{b} \leq \overline{b}$. We need to prove that a given pair of t-norm and

weighting function satisfying (28) and (29) has the $u$-property. To do this we will consider directly the properties of

$$\frac{m + f(t(a,b))}{M + f(b)}\,,$$

which is denoted further by $r(b)$. Let us determine the monotonicity of $r(b)$ separately in both intervals $(\underline{b}, q]$ and $[q, 1]$.

In the first part we consider the following transformation:

$$\begin{aligned} r(b) - r(\underline{b}) &= \frac{m + f(t(a,b))}{M + (b)} - \frac{m + f(t(a,\underline{b}))}{M + f(\underline{b})} \\ &= \frac{f(b) - f(\underline{b})}{f(b) + M}\left[\frac{f(t(a,b)) - f(t(a,\underline{b}))}{f(b) - f(\underline{b})} - \frac{m + f(t(a,\underline{b}))}{M + f(\underline{b})}\right]. \end{aligned} \tag{A.11}$$

By (28), this shows that in the interval $(\underline{b}, q]$

$$r(b) = \lambda_1 h(b) + \lambda_2\,, \tag{A.12}$$

where

$$h(b) = \frac{f(b) - f(\underline{b})}{f(b) + M} \tag{A.13}$$

and $\lambda_1$ and $\lambda_2$ are constants defined as

$$\lambda_1 = \alpha - \frac{m + f(t(a,\underline{b}))}{M + f(\underline{b})} = \alpha - r(\underline{b})\,, \tag{A.14}$$

$$\lambda_2 = \frac{m + f(t(a,\underline{b}))}{M + f(\underline{b})} = r(\underline{b})\,. \tag{A.15}$$

Observe that $h(b)$ is always increasing, thus in this case $r(b)$ is decreasing if and only if $\lambda_1 < 0$ and is non decreasing otherwise. Observe that $\lambda_1$ depends on both $m$ and $M$.

On the other hand $r(b)$ in the interval $(q, 1]$ is also monotonic. To simplify further our considerations, we denote $f(t(a,b))$ by $t_{f,a}(b)$. From the assumptions we know that $t_{f,a}$ is differentiable in $(q, 1]$, and so is $r(b)$:

$$r'(b) = \frac{(M + f(b))\, t'_{f,a}(b) - (m + t_{f,a}(b))\, f'(b)}{(M + f(b))^2}\,. \tag{A.16}$$

To check whether $r(b)$ is increasing or decreasing, we need to determine when its derivative is greater or less than 0. After routine transformations we find that $r(b)$ is increasing whenever

$$\frac{t'_{f,a}(b)}{f'(b)} \geq \frac{m + t_{f,a}(b)}{M + f(b)} = r(b)\,, \tag{A.17}$$

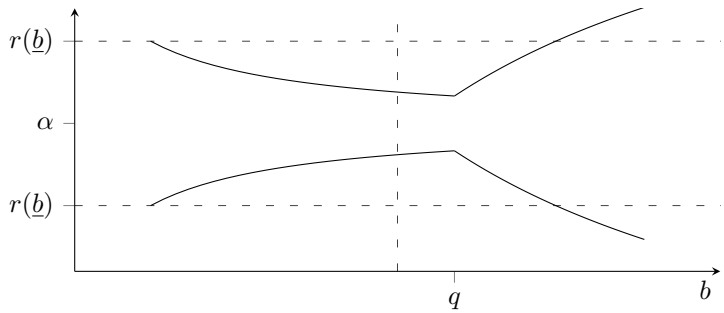


Figure A.7: Shape of the $r(b)$ in second (lower line) and third (upper line) case.

and decreasing whenever

$$\frac{t_{f,a}(b)}{f'(b)} \leq \frac{m + t_{f,a}(b)}{M + f(b)} = r(b)\,. \tag{A.18}$$

Because $r(b)$ always lies between 0 and 1, we can conclude that $r(b)$ is increasing when

$$\frac{t'_{f,a}(b)}{f'(b)} \geq 1 \tag{A.19}$$

and decreasing when

$$\frac{t'_{f,a}(b)}{f'(b)} = 0 \tag{A.20}$$

which because $f$ is increasing reduces further to

$$t'_{f,a}(b) = 0\,. \tag{A.21}$$

This corresponds to the second assumption of the theorem (29). So as a consequence, we know that $r(b)$ is increasing or decreasing in $(q, 1]$.

Finally we have got four options:

1. $r(b)$ is not decreasing in the $(\underline{b}, q]$ and increasing in the $(q, 1]$,
2. $r(b)$ is not decreasing in the $(\underline{b}, q]$ and decreasing in the $(q, 1]$,
3. $r(b)$ is decreasing in the $(\underline{b}, q]$ and increasing in the $(q, 1]$,
4. $r(b)$ is decreasing in the $(\underline{b}, q]$ and $(q, 1]$.

We need to prove that values of arg min and arg max may achieve only two possible values when $m$ and $M$ change. Of course, those values may still depend on other parameters such as $a$ or $\underline{b}$. In the first and last case it is easy to see that minimum and maximum are always attained on boundaries ($r(b)$ is increasing or decreasing in whole its domain). Thus arg min and arg max are always equal to either $\underline{b}$ or $\overline{b}$. The remaining two cases are depicted on Fig A.7. In the second case global minimum may be either in $\underline{b}$ or $\overline{b}$ and global maximum is attained in $q$ or $\overline{b}$ depending whether $q \leq \overline{b}$ or not. Similarly, in the third case global

minimum is attained in $q$ or $\overline{b}$ depending whether $q \leq \overline{b}$ and global maximum may be either in $\underline{b}$ or $\overline{b}$. Observe that in all cases actual location of extrema does not depend on either $m$ or $M$. Hence both $\arg\min$ and $\arg\max$ are equal to either $\underline{b}$ or other value that do not depend on $m$ and $M$. This completes the proof of $u$-property.


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